

## § 2.8 Complex Representation of Waves

It ends up that periodic systems (electrical circuits, mechanical oscillators, quantum waves . . . ) are most easily represented by using complex numbers. We will be using a complex representation for quantum waves. For this reason we are now going to do a short review of complex numbers, in order to remind you (or demonstrate to you for the first time) the properties of complex numbers that we will be useful to our understanding of quantum mechanics.

### Oscillation as a Rotation

Suppose that you have a mass oscillating on the end of a spring. We know that the position of the mass is given as a function of time as

$$x(t) = a \cos \omega t$$

where  $a$  is the amplitude of the oscillation and  $\omega$  is the frequency in units of radians per second ( $\omega = 2\pi/\tau$ , where  $\tau$  is the period of oscillation).

Now consider an object moving clockwise in a circle of radius  $a$  with an angular frequency of  $\omega$ . The position of this object is given by

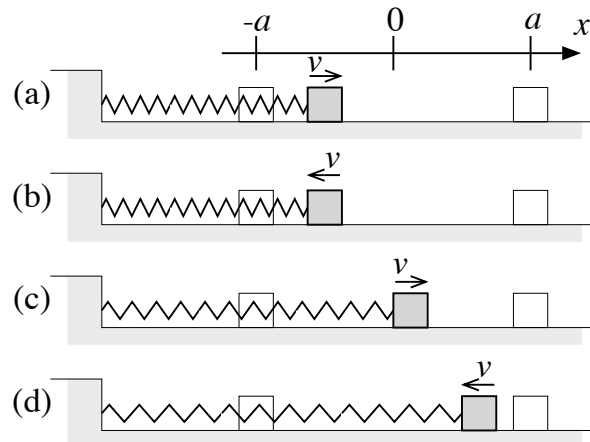
$$x(t) = a \cos \omega t$$

$$y(t) = -a \sin \omega t$$

Thus we find that the horizontal component of a circular motion is the same as the motion of an oscillator. We will find that complex numbers give us a very simple way to represent circular motion and thus a simple way to represent oscillations.

#### ▷ PROBLEM 2.16

Below is shown an oscillator at four different points in its cycle. Draw the corresponding location of a clockwise circular motion for each of the four points, and estimate the angle for each. This angle gives the *phase* of the oscillator in its cycle.



### Definition of a Complex Number

Define a new number  $i$  that has the following property.

$$i^2 = -1$$

This number is called an *imaginary* number since the square of any real number is positive. A *complex* number  $z$  is defined to be a linear combination of imaginary and real parts

$$z = x + iy$$

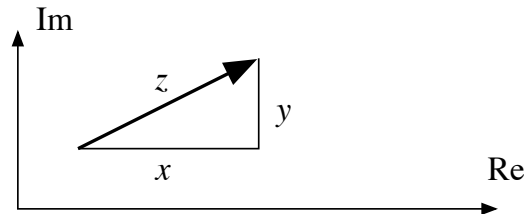
The real numbers  $x$  and  $y$  in the expression above are called the real and imaginary parts of  $z$

$$\Re\{z\} = x$$

$$\Im\{z\} = y$$

### Complex Numbers as Vectors

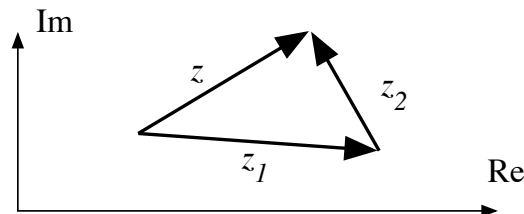
Since a complex number has two components it is often useful to think of a complex number as a vector in two dimensions, with the real and imaginary parts being the  $x$  and  $y$  components of the vector.



It is often useful to visualize algebraic operations in this way. For example, suppose a complex number is the sum of two other complex numbers  $z = z_1 + z_2$  then algebraically we can add the numbers as follows

$$\begin{aligned} z &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

So we find that complex numbers add like vectors, in that one simply adds component by component. Graphically the addition is as follows.



### Multiplication and Polar Coordinates

Let  $z = z_1 z_2$  then

$$\begin{aligned} z &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_1 + x_2 y_1) \end{aligned}$$

so if  $x$  and  $y$  are the components of the product  $z$  then

$$x = x_1x_2 - y_1y_2$$

$$y = x_1y_1 + x_2y_1$$

This is not very intuitive. But if you use polar coordinates to define represent the complex numbers instead then the polar coordinates of the product are related to the polar coordinates of the elements of the product in a simple way.

$$r = r_1r_2$$

$$\theta = \theta_1 + \theta_2$$

As a specific example suppose  $z = z_1z_2$  with

$$z_1 = 2 + 3i \longrightarrow \begin{cases} r_1 = \sqrt{2^2 + 3^2} = 3.606 \\ \theta_1 = \arctan(3/2) = 0.983\text{rad} \end{cases}$$

$$z_2 = 5 + i \longrightarrow \begin{cases} r_2 = \sqrt{5^2 + 1^2} = 5.099 \\ \theta_2 = \arctan(1/5) = 0.197\text{rad} \end{cases}$$

$$z = (2 + 3i)(5 + i) = 10 + 2i + 15i + 3i^2 = 10 - 3 + 17i = 7 + 17i$$

So

$$r = \sqrt{7^2 + 17^2} = 18.385$$

$$\theta = \arctan(17/7) = 1.180\text{rad}$$

and we see that  $r = r_1r_2 = (3.606)(5.099)$  and  $\theta = \theta_1 + \theta_2 = 0.983 + 0.197$ . That the product of complex numbers works in this way can be proved using Euler's Formula

*Theorem: Euler's Formula*

$$e^{i\theta} = \cos \theta + i \sin \theta$$

The standard relationship between rectangular and polar coordinates is

$$x = r \cos \theta$$

$$y = r \sin \theta$$

So that

$$z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

**Theorem: Polar Form of a complex number**

If  $r$  and  $\theta$  are the polar coordinates of a complex number  $z$  then

$$z = re^{i\theta}$$

Euler's formula allows us to write out a complex number in polar coordinates in a very compact notation. It is because of the simplicity of the algebraic form of a complex number in polar coordinates that we use complex numbers to represent period systems.

Using the polar representation of complex numbers we can prove the previously stated result about multiplication.

$$z = z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

**Complex Conjugate**

If  $z = x + iy$  then the complex conjugate of  $z$  is defined to be

$$z^* = x - iy$$

Note the following useful properties.

$$\Re\{z\} = \frac{z + z^*}{2}$$

$$\Im\{z\} = \frac{z - z^*}{2i}$$

$$r^2 = z^* z \equiv |z|^2$$

▷ **PROBLEM 2.17**

Let  $a = 1 + 5i$ , and  $b = 4 - 3i$ .

- Compute  $c = ab$ , the product of  $a$  and  $b$ .
- Write  $a$ ,  $b$ , and  $c$  in polar form.
- Check to see that  $r_c = r_a r_b$  and  $\theta_c = \theta_a + \theta_b$

▷ **PROBLEM 2.18**

Let  $z = 2 + 2i$ . Draw  $z$  as a vector in the complex plane. Now consider the the complex number  $w = e^{i\theta} z$  with  $\theta = \pi/2$ .

- Draw  $w$  in the complex plane.
- How does it compare with  $z$ ?
- Repeat for  $\theta = \pi/4$ ,  $\theta = -\pi/4$  and  $\theta = \pi$ .
- Now generalize. What does multiplying an arbitrary  $z$  by  $e^{i\theta}$  do? Prove your generalization.

▷ **PROBLEM 2.19**

Use Euler's Formula to show the following

- $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
- $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
- $\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$

### Complex Representation of an Oscillator

The position of a general harmonic oscillator can be written as

$$x(t) = a \cos(\phi - \omega t)$$

Where the constant  $\phi$  gives the phase of the oscillator at  $t = 0$ . If we let  $\xi(t) = ae^{i(\phi - \omega t)}$  then we can write  $x$  in terms of  $\xi$ .

$$x(t) = \Re\{\xi(t)\}$$

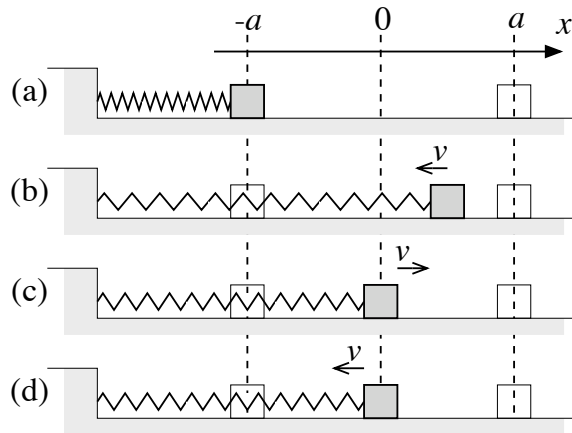
since

$$\begin{aligned} \Re\{\xi(t)\} &= \Re\{ae^{i(\phi - \omega t)}\} \\ &= \Re\{a \cos(\phi - \omega t) + ia \sin(\phi - \omega t)\} \\ &= a \cos(\phi - \omega t) \end{aligned}$$

The function  $\xi(t)$  is the complex representation of the function  $x(t)$ . Note that the complex representation is simply exploiting our observation that an oscillation is a component of a constant velocity rotation. It is good to keep this in mind.

#### ▷ PROBLEM 2.20

Below is shown four different oscillators at  $t = 0$ . Write out the complex representation of each oscillator.



### Complex Representation of a Periodic Wave

We have seen that it is possible to represent an oscillator as a constant velocity rotation in the complex plane. Now we would like to do the same for traveling waves. We know this will work, because at each point in space, the value of the field of the wave oscillates. For example, if you are standing in the water with your eyes closed, you would feel the water rising and falling as the waves in the water pass by you. The depth of the water would oscillate. This is also true at any other location in the water. Thus at each location, there is an oscillation. But the different locations do not all oscillate in sync.